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Approximating Fourier transformation of orbital integrals

C.K. Arthur Lim

University of Minnesota, Vincent Hall, 206 Church St. S.E., Minneapolis, MN 55455, USA

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Abstract

In this paper, we are concerned with orbital integrals on a class $\mathcal{C}$ of real reductive Lie groups with non-compact Iwasawa K-component. The class $\mathcal{C}$ contains all connected semisimple Lie groups with infinite center. We establish that any given orbital integral over general orbits with compactly supported continuous functions for a group G in $\mathcal{C}$ is convergent. Moreover, it is essentially the limit of corresponding orbital integrals for its quotient groups in Harish-Chandra's class. Thus the study of orbital integrals for groups in class $\mathcal{C}$ reduces to those of Harish-Chandra's class. The abstract theory for this limiting technique is developed in the general context of locally compact groups and linear functionals arising from orbital integrals. We point out that the abstract theory can be modified easily to include weighted orbital integrals as well. As an application of this limiting technique, we deduce the explicit Plancherel formula for any group in class $\mathcal{C}$.

Keywords: Fourier transform; Fourier inversion; Orbital integral; Plancherel formula; Semisimple; Infinite center; Real reductive Lie group

1. Introduction

Let G be a real reductive Lie group. Let K be a maximal compact subgroup of G and θ be the Cartan involution of G corresponding to K. Let B be a real symmetric bilinear form on the Lie algebra of G. Suppose (G, K, θ, B) satisfy all assumptions so that G

E-mail address: alim@math.arizona.edu.
URL: http://www.math.arizona.edu/~alim/.

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is in Harish-Chandra's class. Fix a Haar measure dx on G. Let $C_c(G)$ be the space of compactly supported continuous functions on G. Let $L^1(G)$ be the space of absolutely integrable functions on G. Fix an element $y \in G$ and let G^y be the centralizer of y in G. Fix a Haar measure dz on G^y. Let $\dot{x}$ denote the equivalence class of x in G/G^y. Let $d\dot{x}$ denote the unique measure on G/G^y such that

$$\int_G f(x)\,dx = \int_{G/G^y} F(\dot{x})\,d\dot{x},$$

where $F(\dot{x}) = \int_{G^y} f(xz)\,dz$ and $f \in C_c(G)$. For $f \in C_c(G)$, we set

$$\Lambda^y(f) = \int_{G/G^y} f\big(xyx^{-1}\big)\,d\dot{x}.$$

Then Λ^y is a distribution on G. We called Λ^y an orbital integral. Note that if y is the identity element, then Λ^y is δ, the Dirac-δ distribution.

Let $\hat{G}$ denote the set of equivalence classes of irreducible unitary representations of G. Let $\pi \in \hat{G}$ and $f \in L^1(G)$. We define the Fourier transform of f at π by $\pi(f) = \int_G f(x)\pi(x)\,dx$. Let $C_c^\infty(G)$ be the space of compactly supported smooth functions on G. Then $\pi(f)$ is of trace class and we set $\hat{f}(\pi) = \mathrm{tr}(\pi(f))$ for each $f \in C_c^\infty(G)$ and $\pi \in \hat{G}$. Let V be the set of functions $\{\hat{f}\colon\ f \in C_c^\infty(G)\}$ on $\hat{G}$. Let Λ be an invariant distribution on G. Then the Fourier inversion of Λ is a linear functional $\mathcal{F}(\Lambda)$ on V such that, for any $f \in C_c^\infty(G)$, $\mathcal{F}(\Lambda)(\hat{f}) = \Lambda(f)$. A central problem in Harmonic analysis is to compute an explicit formula for $\mathcal{F}(\Lambda)$. The corresponding problem for orbital integrals is studied by many; Arthur [1], Barbasch [3], Bouaziz [4], Harish-Chandra [12], Herb [6], and Sally and Warner [5].

Let $\mathcal{C}$ denote the class of real reductive Lie group G with datum (G, K, θ, B) satisfying all assumptions in Harish-Chandra's class except that K is non-compact but contains a finitely generated central discrete subgroup Γ such that K/Γ is compact. In this paper, we are interested in orbital integrals of groups in class $\mathcal{C}$. The groups in class $\mathcal{C}$ includes all universal covering of real reductive linear algebraic groups. In particular, all real connected semisimple Lie groups with infinite center. We show that the orbital integral of a group in class $\mathcal{C}$ is convergent and, in essence, the limit of corresponding orbital integrals of groups in Harish-Chandra's class. As such the study of orbital integrals of groups in class $\mathcal{C}$ reduces to those of Harish-Chandra's class. Specifically, one can deduce the Fourier inversion formula for orbital integral formula for groups in class $\mathcal{C}$ from those for groups in Harish-Chandra's class by the limiting technique described here. We shall see that the abstract theory for the limiting technique is developed in the general context of locally compact groups and linear functionals arising from orbital integrals. This suggests possible applications of such method to other classes of groups. We note that the content of this paper is an extension of results obtained in the author's doctoral dissertation [16].

We outline the contents of this paper now. In Section 2 we give the abstract theory for the limiting technique mentioned above. We shall see that the development of the abstract theory is very simple and can be adjusted to study weighted orbital integrals in groups of class $\mathcal{C}$. As an application to the limiting technique, we compute in Section 3 the explicit Plancherel formula of a group in class $\mathcal{C}$. Of course, our method relies heavily on known results of Harish-Chandra for his class of reductive groups (see [9–11]). Although the explicit Plancherel formula for groups in class $\mathcal{C}$ has been computed by Herb and Wolf [13] and, independently, by Duflo and Vergne [15], it still interesting to note yet another example of the far-reaching consequences of Harish-Chandra's work. Moreover, this computation is sufficient to illustrate how our theory can be applied to deduce the inversion formula for orbital integral for groups in class $\mathcal{C}$.

2. Abstract theory

Throughout this section, G is a locally compact, separable unimodular group unless otherwise stated. We keep all notations of Section 1. Fix an element $y \in G$. Assume that G^y is unimodular. Suppose G contains a sequence of central discrete subgroups Γ_j, $j \in \mathbb{N}$, such that

(1) $\Gamma_j \subset \Gamma_1$ for all $j \in \mathbb{N}$.
(2) For each compact set $C \subset G$ there exists $N \in \mathbb{N}$, dependent only on C, so that $C \cap \Gamma_j = \emptyset$ for all $j \geqslant N$ or $C \cap \Gamma_j = \{e\}$ for all $j \geqslant N$, where e denotes the identity in G.

Set $G_j = G/\Gamma_j$. Let x_j denote the equivalence class of x in G_j. Define the surjective linear map $\Phi_j : C_c(G) \to C_c(G_j)$ by, for each $f \in C_c(G)$,

$$\Phi_j(f)(x_j) = \sum_{\gamma \in \Gamma_j} f(x\gamma).$$

Let dx_j be the Haar measure on G_j such that

$$\int_G f(x)\, dx = \int_{G_j} \Phi_j(f)(x_j)\, dx_j.$$

Equip $C_c(G)$ and $C_c(G_j)$ with the usual convolution and the involution defined by $f^*(x) = \overline{f(x^{-1})}$ for any function f on G. For each $\pi \in \hat{G}_j$, we may identify π with $\tilde{\pi} \in \hat{G}$ where $\tilde{\pi}(x) = \pi(x_j)$ for any $x \in G$. The following are important but easy-to-verify properties of Φ_j.

Proposition 2.1. *Let $f, h \in C_c(G)$, $\pi \in \hat{G}_j$.*

(1) *Φ_j preserves convolution and involution, that is $\Phi_j(f * h) = \Phi_j(f) * \Phi_j(h)$ and $\Phi_j(f^*) = \Phi_j(f)^*$.*
(2) $\pi(\Phi_j(f)) = \tilde{\pi}(f)$.

Define the orbital integral $\Lambda^y(f)$ with $f \in C_c(G)$ by

$$\Lambda^y(f) = \int_{G/G^y} \big(xyx^{-1}\big)\, d\dot{x}.$$

Note that $G/G^y = G_j/G_j^{y_j}$. Thus we may define the orbital integrals, for each $f \in C_c(G_j)$,

$$\Lambda^{y_j}(f) = \int\limits_{G/G^y} f\big(x_j y_j x_j^{-1}\big)\, d\dot{x}.$$

At this point, we do not know the convergence of any of the above orbital integrals. However, we shall show that $\Lambda^y(f)$ is convergent whenever $\Lambda^{y_j}(\Phi_j(f))$ are convergent for every $j \in \mathbb{N}$. For this purpose, we shall assume that $\Lambda^{y_j}(f)$ is convergent for all $f \in C_c(G_j)$. Thus Λ^{y_j} is a linear functional on $C_c(G_j)$. Moreover, Λ^{y_j} lifts to a linear functional Λ_j^y on $C_c(G)$ defined by, for each $f \in C_c(G)$, $\Lambda_j^y(f) = \Lambda^{y_j}(\Phi_j(f))$.

Lemma 2.2. *Fix $f \in C_c(G)$. Then there exists $N \in \mathbb{N}$, dependent on f and y, such that*

(1) *if $y \in \operatorname{supp}(f)$ then $y_j \cap \operatorname{supp}(f) = \{y\}$ for all $j \geqslant N$;*
(2) *if $y \notin \operatorname{supp}(f)$ then $y_j \cap \operatorname{supp}(f) = \emptyset$ for all $j \geqslant N$.*

Consequently, $\lim_{j\to\infty} \Phi_j(f)(y_j) = f(y)$.

Proof. Take $C = y^{-1} \cdot \operatorname{supp}(f)$. Then C is a compact subset of G. The result follows from the properties of $\{\Gamma_j\}_{j\in\mathbb{N}}$. $\square$

Theorem 2.3. *For each fixed $f \in C_c(G)$ we have $\Lambda^y(f) = \lim_{j\to\infty} \Lambda_j^y(f)$. Moreover, Λ^y is a linear functional on $C_c(G)$.*

Proof. Set $F(\dot{x}) = f(xyx^{-1})$ and $F_j(\dot{x}) = \Phi_j(f)(x_j y_j x_j^{-1})$ for each $x \in G$. As Λ_j^y are linear functionals, $F_j \in L^1(G/G^y)$ for all $j \in \mathbb{N}$. By Lemma 2.2, for each $x \in G$, $\lim_{j\to\infty} F_j(\dot{x}) = F(\dot{x})$. It suffices to consider positive valued f as we may write $f = \operatorname{Re}(f)^+ - \operatorname{Re}(f)^- + i\operatorname{Im}(f)^+ - i\operatorname{Im}(f)^-$ where $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$ denote the real and imaginary parts of f respectively and $i = \sqrt{-1}$. Therefore we have $F_1(\dot{x}) \geqslant F_j(\dot{x}) \geqslant 0$ for all $j \in \mathbb{N}$. Thus by Lebesgue's dominated convergence theorem, $F \in L^1(G/G^y)$. Moreover, we have

$$\lim_{j\to\infty} \Lambda_j^y(f) = \lim_{j\to\infty} \int\limits_{G/G^y} F_j(\dot{x})\, d\dot{x} = \int\limits_{G/G^y} F(\dot{x}) d\dot{x} = \Lambda^y(f). \qquad \square$$

2.1. Real reductive group case

Suppose that G is a real reductive Lie group in class $\mathcal{C}$ with datum (G, K, θ, B). Let Γ be the finitely generated central discrete subgroup in K such that K/Γ is compact. Without loss of generality, we may assume that Γ is torsion free. Let r denote the free rank of Γ and let $\gamma_1, \ldots, \gamma_r$ be the generators of Γ. Let Γ_j denote the central discrete subgroup generated by $\gamma_1^j, \ldots, \gamma_r^j$ for $j \in \mathbb{N}$. Then the group G and its subgroups Γ_j, $j \in \mathbb{N}$, satisfy the assumptions at the beginning of Section 2. Observe that $K_j = K/\Gamma_j$ is compact because Γ/Γ_j is finite and $K/\Gamma = (K/\Gamma_j)/(\Gamma/\Gamma_j)$ is compact. Furthermore,

Γ_j is a central discrete subgroup of G thus $G_j = G/\Gamma_j$ shares the same Lie algebra as G. Consequently, the datum (G_j, K_j, θ, B) satisfies all assumptions in Harish-Chandra's class (see [7, Chapter VII, Section 2]). Thus, G_j is in Harish-Chandra's class and K_j is maximal compact in G_j (see [7, Proposition 7.19]). Moreover, for any $j \in \mathbb{N}$, the integrals Λ_j^y are convergent for all $y \in G$ and are distributions (see [17]). Consequently, $\Lambda^y = \lim_{j\to\infty} \Lambda_j^y$ is a distribution. We may then expressed Theorem 2.3 in terms of the Fourier transform of Λ^y and Λ_j^y. We state it here as a corollary of Theorem 2.3.

Corollary 2.4. *For $f \in C_c^\infty(G)$, $\lim_{j\to\infty} \mathcal{F}(\Lambda_j^y)(\hat{f}) = \mathcal{F}(\Lambda^y)(\hat{f})$.*

We conclude this section by indicating that the above theory can also be applied to the study of weighted orbital integrals (see [2]). Weighted orbital integrals are generalization of orbital integrals discussed above. Moreover, they are in general non-invariant distributions. We shall keep the above notations. The group G is in class $\mathcal{C}$.

Let v be a non-negative measurable function on G/G^y. Define the weighted orbital integral L^y on $C_c(G)$ by, for $f \in C_c(G)$,

$$L^y(f) = \int\limits_{G/G^y} f\left(xyx^{-1}\right)v(\dot{x})\,d\dot{x}.$$

As above, we may also define the weighted orbital integrals on $C_c(G_j)$ by, $f \in C_c(G_j)$,

$$L^{y_j}(f) = \int\limits_{G/G^y} f\left(x_j y_j x_j^{-1}\right)v(\dot{x})\,d\dot{x}.$$

We assume that $L^{y_j}(f)$ is convergent for $f \in C_c(G_j)$. Set $L_j^y(f) = L^{y_j}(\Phi_j(f))$ for $f \in C_c(G)$. Then L_j^y is a linear functional on $C_c(G)$. By the same arguments as in the proof of Theorem 2.3, we obtain the following theorem.

Theorem 2.5. *For each fixed $f \in C_c(G)$ we have $L^y(f) = \lim_{j\to\infty} L_j^y(f)$. Moreover, L^y is a linear functional on $C_c(G)$.*

3. Plancherel formula for groups in class $\mathcal{C}$

In this section, we shall apply the limiting technique developed in Section 2 to compute the Plancherel formula for any group in class $\mathcal{C}$. We begin by setting up notations required for computations later and recall a simple extension of the Peter–Weyl theorem. We will also adopt all notations of Section 2.1.

Let $\mathrm{Car}(G)$ be a complete set of θ-stable Cartan subgroup of G. Let $\mathfrak{g}$ denote the Lie algebra of G. Let $\mathfrak{k}$ denote the Lie algebra of K. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. Let $H \in \mathrm{Car}(G)$ and let $\mathfrak{h}$ denote its Lie algebra. Let $H_R = \exp(\mathfrak{h} \cap \mathfrak{p})$ where $\exp : \mathfrak{g} \to G$ is the usual exponential map. Let $H_I = K \cap H$. Then $H = H_I \cdot H_R$. Note that $\mathrm{Car}(G_j) = \{H/\Gamma_j\colon\ H \in \mathrm{Car}(G)\}$.

3.1. Reduced dual and properties of tempered characters

The reduced dual (or tempered dual) of G is obtained as follows. Let $H \in \mathrm{Car}(G)$. Write $H = H_I \cdot H_R$. For simplicity, set $T = H_I$ and $A = H_R$. Let T^* and A^* denote the set of irreducible unitary characters of T and A, respectively. Let P be a parabolic subgroup containing A. Write $P = MAN$ where M is the Levi subgroup. Then T is a Cartan subgroup of M. Moreover, T/Γ is compact. Thus M has relative discrete series (see [14]) and they are parametrized by $T^{*\prime}$, the set of regular elements of T^*. Fix $\chi \in T^{*\prime}$, let π_χ denote the relative discrete series representation of M corresponding to χ. Let $(V, \langle\,,\rangle)$ be the Hilbert space on which π_χ acts. Fix $\nu \in A^*$. Define the irreducible unitary representation π_χ^ν of P, for $m \in M$, $a \in A$, and $n \in N$, by $\pi_\chi^\nu(man) = \pi_\chi(m)\nu(a)$. Let $\mathcal{H}$ be the space of all V-valued measurable functions on G satisfying, for $f \in \mathcal{H}$,

(1) $f(px) = \pi_\chi^\nu(p)(f(x))$ for all $p \in P$ and $x \in G$.
(2) $\int_{K/\Gamma} \langle f(k), f(k)\rangle \, d\bar{k} < \infty$.

Here $k \mapsto \bar{k}$ is the canonical projection of K onto K/Γ and $d\bar{k}$ is a Haar measure on K/Γ. Endow $\mathcal{H}$ with the inner product $(f, g) = \int_{K/\Gamma} \langle f(k), g(k)\rangle \, d\bar{k}$ for any $f, g \in \mathcal{H}$. Then $\mathcal{H}$ is a Hilbert space. Define the unitary representation $\pi_{\chi,\nu}$ of G in $\mathcal{H}$ by $(\pi_{\chi,\nu}(f)(y))(x) = f(xy)$ for any $x, y \in G$ and $f \in \mathcal{H}$. We shall denote $(\pi_{\chi,\nu}, \mathcal{H})$ by $\mathrm{Ind}_P^G(\pi_\chi^\nu)$. Note that $\pi_{\chi,\nu}$ are tempered for all $\chi \in T^{*\prime}$ and $\nu \in A^*$. Moreover, they are irreducible whenever $\nu \in A^{*\prime}$, the set of regular elements in A^*. For singular element $\nu \in A^*$, $\mathrm{Ind}_P^G(\pi_\chi^\nu)$ is reducible and decomposes into a direct sum of irreducible representations of G. These irreducible representations are called limits of relative discrete series. The reduced dual consists of the collection of all equivalent classes of irreducible representations of G obtained from the family $\{\mathrm{Ind}_P^G(\pi_\chi^\nu)\colon \chi \in T^{*\prime},\ \nu \in A^*\}$. The characters $\Theta(\chi, \nu)$ of the representations $\mathrm{Ind}_P^G(\pi_\chi^\nu)$ are eigendistributions on G.

We further note that each Γ_j-invariant characters in T^* induces a irreducible unitary character on T/Γ_j. In fact, the set of Γ_j-invariant characters in T^* corresponds exactly to $(T/\Gamma_j)^*$. Of course, each Γ_j-invariant $\chi \in T^{*\prime}$ gives a Γ_j-invariant representation $\mathrm{Ind}_P^G(\pi_\chi^\nu)$ for any $\nu \in A^*$, and thus induces a tempered unitary representation on G_j. The collection of all equivalence classes of irreducible representations obtained from all Γ_j-invariant $\mathrm{Ind}_P^G(\pi_\chi^\nu)$ gives the reduced dual of G_j.

Next, we fix a natural measure on T^* which will be used later in our computation of Fourier inversion formula. We recall that $\Gamma \cong \mathbb{Z}^r$ thus $\Gamma^* \cong \mathbb{T}^r$. Here $\mathbb{T}$ is the circle group. For each $\zeta \in \Gamma^*$, we set T_ζ^* to be the collection of all $\chi \in T^*$ such that $\chi(\gamma t) = \zeta(\gamma)\chi(t)$ for all $\gamma \in \Gamma$ and $t \in T$. Then by of [14, Section 2.4], T_ζ^* is discrete and we may write T^* as the disjoint union of T_ζ^*, $\zeta \in \Gamma^*$. Of course, we may also write $(T/\Gamma_j)^*$ as the disjoint union of T_ζ^*, $\zeta \in (\Gamma/\Gamma_j)^*$. Let $d\zeta$ be the Haar measure on Γ^* normalized such that the total volume of Γ^* is one. Let $d\chi$ be the measure on T^* such that, for any $F \in C_c(T^*)$,

$$\int_{T^*} F(\chi)\, d\chi = \int_{\Gamma^*} \sum_{\chi \in T_\zeta^*} F(\chi)\, d\zeta.$$

Extend $\Theta(\chi,\nu)(f)$ to a function on $T^*\times A^*$ by setting $\Theta(\chi,\nu)(f)=0$ whenever χ is singular in T^*. By [8, Lemma 68 of Section 29], we see that, for any polynomial p on $T^*\times A^*$, the function $|p(\chi,\nu)\Theta(\chi,\nu)(f)|$ is bounded. Let $d\nu$ be a Lebesgue measure on A^*. Note that the set of singular points in $T^*\times A^*$ has zero measure with respect to the measure $d\chi\,d\nu$ on $T^*\times A^*$. From above $T^*=\bigcup_{\zeta\in\Gamma^*}T^*_\zeta$, thus we may define the function Ψ on Γ^* by

$$\Psi:\zeta\longmapsto\sum_{\chi\in T^*_\zeta}\int_{A^*}p(\chi,\nu)\Theta(\chi,\nu)(f)\,d\nu.$$

Then Ψ is continuous (see character formula given in [14, Proposition 4.3.10]) except on a set of measure zero. The topology on Γ^* is the usual topology on a product of circle groups.

3.2. Plancherel formula

Normalize the measure dx on G so that dx_1 is the standard Haar measure on G_1 (see [11, Section 7]). We call dx on G the Γ-standard Haar measure. With dx fixed, we obtained the induced measures dx_j on G_j defined in Section 2. Let dk_j denote the measure of K_j. Note that K_j is a j^r-fold cover of K_1. Moreover, the volume of K_1 is 1 under the standard Haar measure for G_1. Consequently, $\int_{K_j}dk_j=j^r$ and dx_j is j^r times the standard Haar measure on G_j. Fix $f\in C_c^\infty(G)$. Then the Plancherel formula of G_j associated to the Haar measure dx_j, according to Harish-Chandra in [11], is given by

$$\Phi_j(f)(e_j)=\sum_{T\cdot A\in\mathrm{Car}(G)}\frac{C(G_j/A)}{j^r}\sum_{\chi\in(T/\Gamma_j)^*}d(\chi)\times\int_{A^*}\mu(A:\chi:\nu)\,\mathrm{Tr}\big(\pi_{\chi,\nu}\big(\Phi_j(f)\big)\big)\,d\nu.$$

Here $C(G_j/A)=c(G_j/A)^{-2}\gamma(G_j/A)^{-1}[\mathfrak{w}(G_j/A)]^{-1}$ where $c(G_j/A)$ (see [11, Section 11]), $\gamma(G_j/A)$ (see [11, Lemma 2.6]) and $[\mathfrak{w}(G_j/A)]$ (see [11, p. 168]) are constants. From the definition of these constants, we see that they are independent of j because G_j is a quotient of G by a central discrete subgroup contained in K. We set $C(G/A)=C(G_j/A)$. The positive number $d(\chi)$ is the formal degree of the discrete series representation π_χ whenever χ is regular and zero otherwise. The function $\mu(A:\chi:\nu)$ is the density function for the Plancherel measure. By Proposition 2.1, we have $\mathrm{Tr}(\pi_{\chi,\nu}(\Phi_j(f)))=\Theta(\chi,\nu)(f)$. We shall identify $(T/\Gamma_j)^*$ with the set of Γ_j-invariant characters in T^*. Thus we may rewrite the above formula as

$$\Phi_j(f)(e_j)=\sum_{T\cdot A\in\mathrm{Car}(G)}\frac{C(G/A)}{j^r}\sum_{\zeta\in(\Gamma/\Gamma_j)^*}\sum_{\chi\in T^*_\zeta}d(\chi)\times\int_{A^*}\mu(A:\chi:\nu)\Theta(\chi,\nu)(f)\,d\nu.$$

From the formula of $d(\chi)$ (see [11, Section 23]), we see that $\chi \mapsto d(\chi)$ is a continuous function on T^*. The function $\mu(A:\chi:\nu)$ is part of the density function for the Plancherel measure. From the computation in [11], one can see that $\mu(A:\chi:\nu)$ defines a positive valued continuous function on $T^* \times A^*$. In fact, as pointed out in [11], $\mu(A:\chi:\nu)$ is a product of polynomials and hyperbolic type functions that appear in $SL(2,\mathbb{R})$ computations. Thus $\mu(A:\chi:\nu)$ naturally extends to a continuous function on the space $T^* \times A^*$ which we would still denote by $\mu(A:\chi:\nu)$. We note that $d(\chi)\mu(A:\chi:\nu)$ has growth bounded by some polynomial on $T^* \times A^*$. Thus, as explained in Section 3.1, the following map is continuous on T^* except on a set of measure zero:

$$\zeta \mapsto \sum_{\chi \in T^*_\zeta} d(\chi) \int_{A^*} \mu(A:\chi:\nu)\Theta(\chi,\nu)(f)\,d\nu.$$

Letting j tend to infinity, we get

$$f(e) = \sum_{T\cdot A \in \mathrm{Car}(G)} C(G/A) \int_{\zeta \in \Gamma^*} \sum_{\chi \in T^*_\zeta} d(\chi) \int_{A^*} \mu(A:\chi:\nu)\Theta(\chi,\nu)(f)\,d\nu\,d\zeta.$$

Thus, by the uniqueness of the Plancherel measure and Theorem 2.3, the Plancherel formula of G associated to the Haar measure dx is given by

$$f(e) = \sum_{T\cdot A \in \mathrm{Car}(G)} C(G/A) \int_{T^*} d(\chi) \int_{A^*} \mu(A:\chi:\nu)\Theta(\chi,\nu)(f)\,d\nu\,d\chi.$$

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References

[1] J. Arthur, Harmonic analysis of invariant distributions, in: Lie Theories and Their Applications, Proc. Ann. Sem. Canad. Math. Congr., Queen's Univ., Kingston, Ontario, 1977, in: Queen's Papers in Pure and Appl. Math., vol. 48, Queen's Univ., Kingston, Ontario, 1978, pp. 384–393.

[2] J. Arthur, On the transfer of distributions: weighted orbital integrals, Duke Math. J. 99 (1999) 209–283.

[3] D. Barbasch, Fourier inversion for unipotent invariant integrals, Trans. Amer. Math. Soc. 249 (1979) 51–83.

[4] A. Bouaziz, Formule d'inversion des intéegrales orbitales sur les groupes de Lie réductifs (Inversion formula for orbital integrals on reductive Lie groups), J. Funct. Anal. 134 (1995) 100–182 (in French).

[5] P.J. Sally Jr., G. Warner, The Fourier transform on semisimple Lie groups of real rank one, Acta Math. 131 (1973) 1–26.

[6] R.A. Herb, Fourier inversion and the Plancherel theorem of semisimple real lie groups, Amer. J. Math. 104 (1982) 9–58.

[7] A.W. Knapp, Lie Groups Beyond an Introduction, second ed., in: Progr. Math., vol. 140, Birkhäuser Boston, Boston, MA, 2002.
[8] Harish-Chandra, Discrete series for semisimple Lie groups. I. Construction of invariant eigendistributions, Acta Math. 113 (1965) 241–318.
[9] Harish-Chandra, Harmonic analysis on real reductive groups. I. The theory of the constant term, J. Funct. Anal. 19 (1975) 104–204.
[10] Harish-Chandra, Harmonic analysis on real reductive groups. II. Wavepackets in the Schwartz space, Invent. Math. 36 (1976) 1–55.
[11] Harish-Chandra, Harmonic analysis on real reductive Lie groups III, Ann. of Math. 104 (1976) 117–201.
[12] Harish-Chandra, Supertempered Distributions on Real Reductive Groups, in: Adv. Math. (Suppl. Stud.), vol. 8, Academic Press, New York, 1983, pp. 139–153.
[13] R. Herb, J.A. Wolf, The Plancherel theorem for general semisimple Lie groups, Compositio Math. 57 (1986) 271–355.
[14] J.A. Wolf, The Action of a Real Semisimple Lie Group on a Complex Flag Manifold. II. Unitary Representations on Partially Holomorphic Cohomology Spaces, in: Mem. Amer. Math. Soc., vol. 138, Amer. Math. Soc., Providence, RI, 1974.
[15] M. Duflo, M. Vergne, La Formule de Plancherel des Groupes de Lie Semi-simples réels, in: Representations of Lie Groups, Kyoto, Hiroshima, 1986, in: Adv. Stud. Pure Math., vol. 14, Academic Press, Boston, MA, 1988, pp. 289–336.
[16] C.K. Arthur Lim, Plancherel formula for connected semisimple Lie groups with infinite center, PhD dissertation, Department of Mathematics, University of Utah, 2001.
[17] R. Ranga Rao, Orbital integrals in reductive groups, Ann. of Math. (2) 96 (1972) 505–510.